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Introduction to string field theory

References:
- SFT – A modern introduction (Springer '21)
- A field theory approach to string theory (Habilitation thesis)

In a nutshell:
- worldsheet string theory = first-quantized theory
  → various problems, need field theory (second quantization)
- SFT is formulated in ket representation
  → consider first scalar field QFT for intuition
- second-quantized ket representation:
  - reparametrization invariance
    → first-quantized Hamiltonian $= 0$ (on-shell condition)
    → second-quantized free eom
    → free action + propagator
  - decompose amplitudes: graphs with propagators + fundamental vertex
    ↳ interactions in action
- string theory: above procedure has also geometric interpretation (for Riemann surfaces)

Outline:
1. Motivations
2. Scalar field theory in the SFT language
3. SFT from reverse engineering of worldsheet

Motivations

Standard model of particle physics / general relativity:
- fundamental quanta = point particles
- simplest approach: classical / quantum mechanics
  (worldline / 1st quantized)
  = describe particle embedded in spacetime
- modern language: field theory (2nd quantized)
  = describe particles as oscillations of field filling all spacetime

String theory: same evolution
- start with worldsheet description
  could go far thanks to properties of 2d theories (symmetries, CFT, Riemann surfaces)
- problems lead to find a field theory
  bonus: can get intuition on string theory from QFT only

Limitations of worldsheet description:
- on-shell
  * divergences associated with on-shell internal particles
    solution: renormalization ⟵ off-shell
    (mass renormalization, vacuum shift)
  * consistency properties (unitarity, crossing sym., analyticity ...)
- no potential to minimize for finding vacua = class. sol.
  need consistent backgrounds to compactify from $D=10$ to $D=4$
- perturbative
  ⟶ non-perturbative, collective, thermal effects?
- background dependence

Main research areas:

1. construction and structure of the action
   → formal and general properties
   → how to make explicit computations?
2. consistency of string theory
3. improvement of worldsheet computations
4. classical solutions (open SFT)
   note: Witten's SFT has very peculiar properties making this possible, but not useful to understand SFT in general

Review: 1703.06410, book (Springer '21)

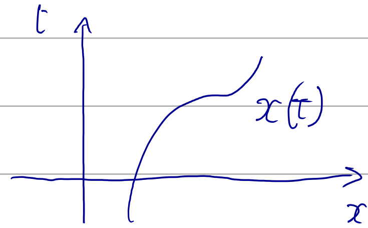
Goals: study effective SFT
- focus on low-energy effective action ("massless" modes)
  → ignore ∞ number of "massive" fields a priori not useful for pheno.
  → try to remove (some) off-shell data
- understand general structure (symmetries, observables)
- deform action with source
- compute higher-derivative corrections

Related work:
- Koyama - Okawa - Suzuki: 2006.16449, 2006.16710
- Maccaferri - Merlano: 1801.07607, 1905.04958

Scalar field theory

SFT may look esoteric (not using standard QFT language)
$\longrightarrow$ consider toy model for making contact

What is a scalar field?



$$\varphi(x) : \mathbb{R}^d \longrightarrow \mathbb{R}$$

$\hookrightarrow$ worldline: $x^\mu(\tau)$ at fixed proper time τ

$$\frac{d\phi}{d\tau} = H\phi = 0 \quad \text{(reparametrization inv.)}$$

What are operations on fields? just multiplication

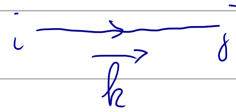
N real scalar fields φ_i (momentum space, Euclidean):

$$S = \frac{1}{2}\sum_{i=1}^{N}\int \frac{d^Dk}{(2\pi)^D}\,\varphi_i(k)\,(k^2+m_i^2)\,\varphi_i(-k)$$
$$+\frac{1}{3!}\int\prod_{i=1}^{3}\frac{d^Dk_i}{(2\pi)^D}\,V^{(3)}_{i_1i_2i_3}(k_1,k_2,k_3)\,\varphi_{i_1}(k_1)\,\varphi_{i_2}(k_2)\,\varphi_{i_3}(k_3)$$
$$+\frac{1}{4!}\int\prod_{i=1}^{4}\frac{d^Dk_i}{(2\pi)^D}\,V^{(4)}_{i_1\cdots i_4}(k_1,\dots,k_4)\,\varphi_{i_1}(k_1)\cdots\varphi_{i_4}(k_4)$$

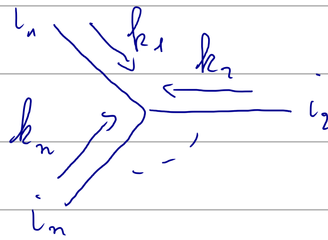
[Weinberg, vol. 2, sec. 21.6]

φ^4 theory $(N=1)$: $V^{(3)}=0$, $V^{(4)}(k_1,\dots,k_4) = \lambda\,\delta^{(D)}(k_1+\dots+k_4)$

Feynman rules



$$= \Delta_{ij}(k) = \frac{\delta_{ij}}{k^2+m_i^2}$$

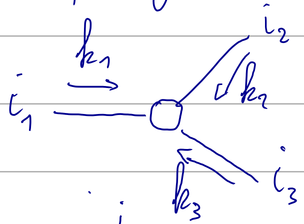


$$= -V^{(n)}_{i_1\cdots i_n}(k_1,\dots,k_n) \qquad n=3,4$$

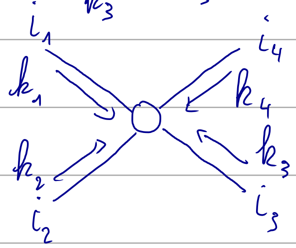
Amplitudes

Imagine you get the amplitudes $A^{(3)}$, $A^{(4)}$ from some method
(scattering amplitude program, worldsheet path integral...)

$$A^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3) =$$



$$A^{(4)}_{i_1 \dots i_4}(k_1, \dots, k_4) =$$



$\longrightarrow$ reverse engineering of vertices if propagator known
↳ action

Strategy:
1. get propagator from first quantized theory
2. factorize amplitude

Ket representation

Introduce complete basis of first-quantized states

$$\mathcal{H} = \text{Span}\{|\phi_i(k)\rangle\}$$

eigenstates of momentum operator p, mass operator m

$$p|\phi_i(k)\rangle = k|\phi_i(k)\rangle \quad , \quad m|\phi_i(k)\rangle = m_i|\phi_i(k)\rangle$$

Define an inner product: non-degenerate

$$\langle\phi_i(k)|\phi_j(k')\rangle = (2\pi)^D \delta_{ij}\, \delta^{(D)}(k+k')$$

(but spacetime momentum repr.)

Worldline position representation: operator X conj to p

$$|\phi_i(k)\rangle = \varepsilon_i\, e^{ik\cdot X}|0\rangle = \varepsilon_i|k\rangle \longrightarrow \text{vertex operators}$$

↳ internal "polarization" = $SO(N)$ generators

On-shell states: 0-eigenvalue of worldline Hamiltonian

$$H|\phi_i(k)\rangle = (k^2+m_i^2)|\phi_i(k)\rangle = 0$$

→ field theory eq. of motion (free theory)

Field ket = linear combination of 1st quantized states

$$|\varphi\rangle = \int \frac{d^Dk}{(2\pi)^D}\, \varphi_i(k)\, |\phi_i(k)\rangle$$

↳ original spacetime fields = coefficients

Equivalent to Fourier expansion:

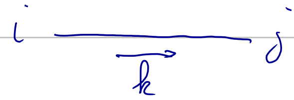
$$\langle x|\phi_i(k)\rangle = \varepsilon_i\langle x|k\rangle = \varepsilon_i\, e^{ikx}$$

$\varphi = T_i\varphi_i$

$$\Rightarrow \varphi(x) = \langle x|\varphi\rangle = \int \frac{d^Dk}{(2\pi)^D}\, \varepsilon_i\, \varphi_i(k)\, e^{ikx}$$

Amplitude factorization

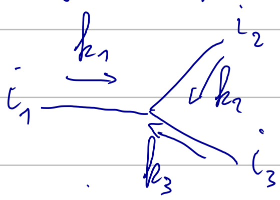
Worldline on-shell condition ⟶ free action ⟶ propagator



$$= \frac{\delta_{ij}}{k^2 + m_i^2}$$

Cubic amplitude: single graph = cubic vertex

$$A^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3) =$$



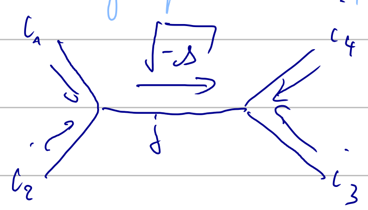
$$= -V^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3)$$

Quartic amplitude: two types of graphs

$$A^{(4)}_{i_1 \dots i_4}(k_1, \dots, k_4) = -V^{(4)}_{i_1 \dots i_4}(k_1, \dots, k_4) + \mathcal{J}^{(4)}_{i_1 \dots i_4}(k_1, \dots, k_4)$$

where $\mathcal{J}^{(4)}$ contains graphs with propagators (s-, t-, u-channels)

$$\mathcal{J}^{(4)}_{i_1 \dots i_4} =$$



+ perms

$$= \sum_j V_{i_1 i_2 j}(k_1, k_2, \sqrt{-s}) \frac{1}{-s + m_j^2} V_{i_3 i_4 j}(k_3, k_4, \sqrt{-s})$$

+ perms

Action in ket representation

Kinetic term: worldline $H : \mathcal{H} \to \mathcal{H}$

Free eom: $H|\varphi\rangle \in \mathcal{H}$ but need $S \in \mathbb{C}$

$\to$ use inner product

$$S_{free} = \frac{1}{2} \langle \varphi | H | \varphi \rangle \equiv \frac{1}{2} \mathcal{V}_2(\varphi^2)$$

Interactions: define vertices $\mathcal{V}_n : \mathcal{H}^{\otimes n} \to \mathbb{C}$ (linear) from action on basis states:

$$\mathcal{V}_n(|\phi_{i_1}(k_1)\rangle, \ldots, |\phi_{i_n}(k_n)\rangle) \equiv V^{(n)}_{i_1, \ldots, i_n}(k_1, \ldots, k_n)$$

$$\equiv \langle \mathcal{V}_n | \phi_{i_1}(k_1) \rangle \otimes \cdots \otimes |\phi_{i_n}(k_n)\rangle$$

Action:

$$S = \frac{1}{2} \mathcal{V}_2(\varphi^2) + \frac{1}{3!} \mathcal{V}_3(\varphi^3) + \frac{1}{4!} \mathcal{V}_4(\varphi^4)$$

$\to$ reproduces original QFT action

Note: $\mathcal{V}_n$ is linear, inner-product non-degenerate

$$\mathcal{V}_n(A_1, \ldots, A_n) \equiv \langle A_1 | \ell_{n-1}(A_2, \ldots, A_n) \rangle$$

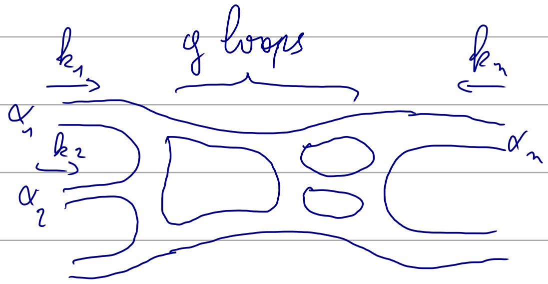
$\hookrightarrow$ interaction products

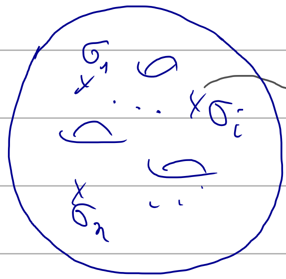
$\ell_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ satisfy L_∞ algebra

$\hookrightarrow$ encodes gauge symmetry, BV structure (= consistent quantization)

Classical string field theory

Worldsheet path integral

$$A_{g,n}(k_1, \dots, k_n)_{\alpha_1, \dots, \alpha_n} = \sum_{\text{surfaces}}$$ 

$$= \sum_{\text{surfaces}}$$ 

vertex op. $V_{\alpha_i}(k_i;\sigma_i)$

genus-g Riemann surface with n punctures

$$= \int \frac{dg_{ab}\, dX}{\text{Vol gauge}} e^{-S_m[g,X]} \left\langle \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{g}\, V_{\alpha_i}(k_i;\sigma_i) \right\rangle$$

$$\overset{\text{gauge fixing}}{=} \int_{\mathcal{M}_{g,n}} d^{M_{g,n}} t \left\langle \text{ghosts} \prod_{i=1}^{n} V_{\alpha_i}(k_i) \right\rangle_{\Sigma_{g,n}(t)}$$

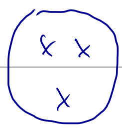
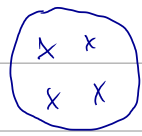
$\langle \dots \rangle \longrightarrow$ 2d CFT correlations functions

$t \longrightarrow$ moduli

$\mathcal{M}_{g,n} \longrightarrow$ moduli space = set of Riemann surfaces which are topologically inequivalent

Note: $A_{g,n}$ often ill-defined $\longrightarrow$ SFT will show why + regularizes

Moduli parameters $t_\lambda \in \mathcal{M}_{g,n} \simeq \{$size of holes, puncture positions$\}$

$\Sigma_{0,3} =$  $\qquad \Sigma_{0,4} =$ 

$\mathcal{M}_{0,3} = 1 \qquad \mathcal{M}_{0,4} = \mathbb{C}$

States, string field and free action

String field: $\Psi[X(\sigma)]$: functional! How to define product?
$\hookrightarrow$ from worldsheet $X(\tau,\sigma)$
$\longrightarrow$ use ket representation, look for vertices $V_n : \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}$

Field ket:

$$|\Psi\rangle = \int \frac{d^Dk}{(2\pi)^D}\, \varphi_\alpha(k)\, |\phi_\alpha(k)\rangle$$

$$= \int \frac{d^Dk}{(2\pi)^D}\Big[\underbrace{T(k)\,|\phi_T(k)\rangle}_{\text{tachyon}} + \underbrace{\varphi_i(k)\,|\phi_i(k)\rangle}_{\text{scalars}} + \underbrace{A^\mu(k)\,|\phi_\mu(k)\rangle}_{U(1)\text{ gauge field}} + \cdots\Big]$$

Interpretation: two Fourier transforms

$$X^\mu(\sigma) \equiv x^\mu + \hat{X}^\mu(\sigma)$$

- center-of-mass $x^\mu \longrightarrow$ continuous momentum k^μ
- $\hat{X}^\mu(\sigma) : \sigma \in I, S^1 \longrightarrow$ discrete modes indexed by α
  $\hookrightarrow$ compact

$$\hat{X}^\mu(\sigma) \sim \sum_n a_n^{+\mu}\, e^{in\sigma}$$

basis: all products of $\{a_n^{+\mu}\}$
other worldsheet structure: $\{a_n^{+i}, \dots\}$

On-shell condition: $H|\Psi\rangle = 0$ (worldsheet Hamiltonian)

2d CFT $\Longrightarrow$ $H = L_0 \equiv \alpha' p^2 + \underbrace{\hat{L}_0}_{\alpha' m^2}$

Action from inner-product (canonical construction in 2d CFT):

$$S_{free} = \frac{1}{2}\langle \Psi | L_0 \Psi\rangle$$

Feynman rules and action

Propagator

$$\frac{1}{L_0} = \int_0^\infty ds\, e^{-sL_0}$$

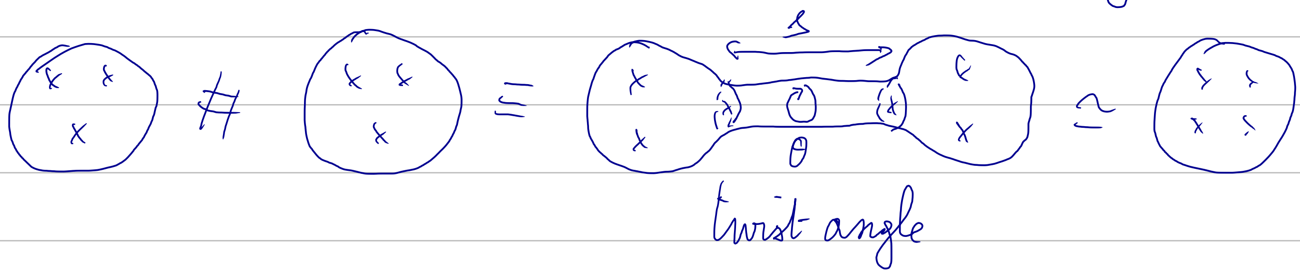
↳ Schwinger parameter

L_0 = generator of dilatation

⟶ e^{-sL_0} inserts a piece of worldsheet of length s

Mathematical operation $\#$ = plumbing fixture

Some $\Sigma_{0,4}$ can be described as two $\Sigma_{0,3}$ connected by a tube:



vary s and θ $\Longrightarrow$ $\mathcal{M}_{0,3} \# \mathcal{M}_{0,3} \equiv \mathcal{F}_{0,4} \subset \mathcal{M}_{0,4}$

missing region $\mathcal{V}_{0,4}$ ≡ fundamental vertex region

Note: why worldsheet amplitudes are ill-defined

- moduli parameters ≃ Schwinger parameters
- Schwinger parametrization divergent if $L_0 = \alpha'(p^2 + m^2) < 0$ (same in QFT)
- SFT uses directly L_0^{-1}, well-defined for $L_0 < 0$

Construct $\tilde{\mathcal{F}}_{g,n}$ and $\mathcal{V}_{g,n}$ recursively.

$$\mathcal{F}_{g,n}(k_1, \dots, k_n)_{\alpha_1 \dots \alpha_n} = \int_{\tilde{\mathcal{F}}_{g,n}} d^{M_{g,n}} t \left\langle \text{ghosts} \times \prod_{i=1}^{n} V_{\alpha_i}(k_i) \right\rangle$$

↳ all surfaces $\Sigma_{g,n}$ obtained by gluing lower-order surfaces

interpreted as sum of Feynman diagrams with propagators

$$\mathcal{V}_{g,n}(k_1, \dots, k_n)_{\alpha_1 \dots \alpha_n} = \int_{\mathcal{V}_{g,n}} d^{M_{g,n}} t \left\langle \text{ghosts} \times \prod_{i=1}^{n} V_{\alpha_i}(k_i) \right\rangle$$

↳ remaining surfaces $\mathcal{V}_{g,n} = \mathcal{M}_{g,n} - \tilde{\mathcal{F}}_{g,n}$

interpreted as fundamental vertices

String field action

$$S = \sum_{g,n \geq 0} \frac{\hbar^{2g} g_s^{2g+n-2}}{n!} \mathcal{V}_{g,n}(\Psi^n)$$

$$= \sum_{g,n \geq 0} \frac{\hbar^{2g} g_s^{2g+n-2}}{n!} \langle \Psi, \underbrace{\ell_{g,n-1}}_{\text{string products}}(\Psi^{n-1}) \rangle$$

In practice: very hard to find all $\mathcal{V}_{g,n}$

note: precise def. of corr. function involve local coordinates on $\Sigma_{g,n}$, which are $\neq$ for $\mathcal{V}_{g,n}$ and $\tilde{\mathcal{F}}_{g,n}$
also very hard

L_∞ relations

(include ghosts)

string field $\Psi \in \mathcal{H}$

$\hookrightarrow$ Hilbert space of string states

Classical SFT action:

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \langle \Psi, V(\Psi) \rangle$$

$$V(\Psi) = \sum_{n \geq 2} \frac{1}{(n+1)!} \ell_n(\Psi^n)$$ (omit index 0: $\ell_{0,n} \to \ell_n$)

Eq. of motion:

$$\mathcal{E}(\Psi) \equiv Q\Psi + V'(\Psi) = 0, \qquad V(\Psi) \equiv \int_0^1 dt\, V'(t\Psi)$$

String products satisfy L_∞ relations: $\ell_1 = Q$

$$0 = \sum_{n_1+n_2=n} \ell_{n_1+1}(A_1, \dots, A_k, \ell_{n_2}(A_{k+1}, \dots, A_n))$$

First relations: $\ell_2(A,B) \equiv [A,B]$

$Q^2 = 0$

$\longrightarrow$ Q nilpotent

$$Q\,\ell_2(A_1, A_2) \pm \ell_2(QA_1, A_2) \pm \ell_2(A_1, QA_2) = 0$$

$\longrightarrow$ Q derivative of ℓ_2

$$Q\,\ell_3(A_1, A_2, A_3) \pm \ell_3(QA_1, A_2, A_3) \pm \dots$$
$$= \pm\, \ell_2(\ell_2(A_1, A_2), A_3) \pm \dots$$

$\longrightarrow$ failure of Q to be a deriv. of ℓ_3 = failure of Jacobi identity